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Algebraic Bethe ansatz for the two species ASEP with different hopping rates

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Abstract
An ASEP with two species of particles and different hopping rates is considered on a ring. Its integrability is proved, and the nested algebraic Bethe ansatz is used to derive the Bethe equations for states with arbitrary numbers of particles of each type, generalizing the results of Derrida and Evans [10]. We also present formulae for the total velocity of particles of a given type and their limit given the large size of the system and the finite densities of the particles.

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1. Introduction

An idea which has proved to be quite useful in understanding the behavior of systems out of equilibrium is to study solvable models. An example of such models, which has been given a lot of attention for at last two decades, is the asymmetric simple exclusion process (ASEP) [1]. It describes a driven lattice gas [2, 3] where particles can hop on adjacent sites with asymmetric rates and hard core exclusion.

Different methods have been applied to study the ASEP, and each of them seems better suited to study certain aspects of the problem. The matrix product ansatz for example has been employed with success in determining the density profile of steady states, steady currents or diffusion coefficients (for reviews see [4–6]). On the other hand other quantities like the relaxation time are more easily dealt with by means of the Bethe ansatz [7, 8].

Actually in [9], Derrida and Lebowitz showed how a modification of the Bethe ansatz of Gwa and Sphon [8] could be used to compute the full large deviation function of the time averaged current for the ASEP with one species of particles.

Soon after Derrida and Evans [10] considered the problem with a second species of particles, and thanks to a Bethe ansatz they were able, not only to reproduce known results [11, 12] about the phase diagram of the steady current of a particle of second type, but also to compute its diffusion coefficient and in principle all the higher cumulants.

Multi-species generalizations of the ASEP have been considered in several papers [13–17]. The fact that they are integrable is not at all obvious. The most natural integrable generalizations of the single species have a hierarchical structure based on quotients of the Hecke algebra, which ensure the integrability, as explained in [18].

On the other hand, in the ASEP with two kinds of particles and different rates, as considered by Derrida and Evans, the hierarchy is partially spoiled precisely by the different hopping rates, and to understand its integrability from a point of view of the Yang–Baxter equation one cannot resort to the Hecke algebra commutation relations.

The proper definition of the model we are going to study goes as follows. We consider a one-dimensional lattice with periodic boundary conditions (i.e. a ring). Each site can be empty (indicated by 0), or occupied either by a particle of type 1 or by a particle of type 2. The rules that govern the stochastic evolution of the system during an interval of time dt are purely local on couples of neighboring sites and are given by

$$\begin{aligned} &10 \to 01 \text{ with rate } 1\\ &20 \to 02 \text{ with rate } \alpha\\ &12 \to 21 \text{ with rate } \beta. \end{aligned}$$

Our first point in the present paper is to make manifest the integrability of the ASEP with two species, by showing an R-matrix which solves the Yang–Baxter equation and gives, through the usual procedure, the transition matrix of our problem. Once we have this we employ the machinery of the algebraic Bethe ansatz (ABA) to derive the Bethe equations and the eigenvalues of the transition matrix (see [19] for a review on ABA, and [20] for recent application of ABA to the ASEP).

Since we have a number of species greater than one we are led to perform a nested Bethe ansatz. In the case of an arbitrary number of species, but with hopping rates independent of the types of particles the nested Bethe equations have already been derived in [21].

With the Bethe equations at our disposal we can tackle the problem of determining the cumulants of the total velocity of particles of the a given type, or of joint cumulants, in the presence of an arbitrary number of particles of each kind.

The exact formula for the average velocity of particles of the second type is given in terms of certain contour integrals

$$F^{\alpha}_{N,M_1,M_2} = \left[\oint_1 + \oint_{1/\alpha}\right] \frac{\mathrm{d}y}{2\pi} \frac{y^N}{(y-1)^{M_1}(\alpha y - 1)^{M_2}}, \tag{1}$$

$$F^{b}_{N,M_1,M_2} = \left[\oint_1 + \oint_{1/\alpha}\right] \frac{\mathrm{d}y}{2\pi} \frac{y^N}{(y-1)^{M_1}(1 - b y)^{M_2}}. \tag{2}$$

The formula for the total velocity of particles of species 2 reads

$$v_2 = M_2 \frac{F^{\alpha}_{N-2,M_1,M_2} F^{b}_{N,M_1+1,M_2} - F^{\alpha}_{N,M_1+1,M_2} F^{b}_{N-2,M_1,M_2}}{F^{\alpha}_{N-1,M_1,M_2} F^{b}_{N,M_1+1,M_2} - F^{\alpha}_{N,M_1+1,M_2} F^{b}_{N-1,M_1,M_2}}, \tag{3}$$

where $b = 1 - \beta$.

We also consider the limit of the large size of the system with finite nonzero densities of particles. In this limit, one can evaluate the contour integrals by saddle point methods finding for the limiting velocity

$$v_2 = M_2 \left(\frac{1}{y^+_{\alpha}} + \frac{1}{y^-_b} - 1 \right), \tag{4}$$

where y_α^+ and y_b^- can be expressed by a unique formula

$$y_\kappa^\pm = \frac{\kappa + 1 - \rho_1 - \kappa\rho_2 \pm \sqrt{(\kappa + 1 - \rho_1 - \kappa\rho_2)^2 - 4\kappa(1 - \rho_1 - \rho_2)}}{2\kappa(1 - \rho_1 - \rho_2)}, \tag{5}$$

where we have to substitute respectively $\kappa = \alpha$ or $\kappa = b$.

The plan of the paper is the following. In section 2, we show the integrability of the ASEP with generic rates by presenting an R-matrix which solves the Yang–Baxter equation and generates the transition matrix of the ASEP. In the same section, we use the techniques of the algebraic Bethe ansatz to diagonalize the transition matrix, arriving at a set of nested Bethe equations. In section 3, we analyze the Bethe equations and derive the exact formula for the total velocity of M_2 particles of type 2 on a ring of size N and in the presence of M_1 particles of kind 1, we comment on the derivation of the higher cumulants. The large-N limit of the velocity is worked out in section 4 where we show that for nonzero densities of particles of each type, its dependence on the parameters α, β and on the densities of the two species of particles ρ_1 and ρ_2 is analytical. In appendix A, we sketch the derivation of the Bethe equations for an ASEP with twisted boundary conditions, that we need for the derivation of the nested equations. In appendix B, we study the integrability of models with higher number of particles and arbitrary hopping rates.

2. Yang–Baxter for two species and different rates

In [10], Derrida and Evans have employed the coordinate Bethe ansatz to study the ASEP in the presence of an impurity, which for us is nothing other than a second species of particles.

The fact that the problem can be solved by Bethe ansatz, as done in [10] for the case of a single second type particle, means that it is integrable. Our first task is to understand better its integrability showing the Yang–Baxter equation behind it.

The transition matrix of our system can be written in terms of matrices that encode the local transition of particles. Let us define the basis of the local space of states as

$$\begin{aligned}
|0\rangle &= \text{empty} \equiv \text{particle of kind } 0,\\
|1\rangle &= \text{ particle of kind } 1,\\
|2\rangle &= \text{particle of kind } 2.
\end{aligned}$$

In the basis $(|0\rangle, |1\rangle, |2\rangle)_a \otimes (|0\rangle, |1\rangle, |2\rangle)_b$ we introduce the operators

$$\begin{aligned}
E^{(10)} &= \mathrm{e}^{\nu_{10}}|0_a, 1_b\rangle\langle 1_a, 0_b| - |1_a, 0_b\rangle\langle 1_a, 0_b|;\\
E^{(20)} &= \mathrm{e}^{\nu_{20}}|0_a, 2_b\rangle\langle 2_a, 0_b| - |2_a, 0_b\rangle\langle 2_a, 0_b|;\\
E^{(12)} &= \mathrm{e}^{\nu_{12}}|2_a, 1_b\rangle\langle 1_a, 2_b| - |1_a, 2_b\rangle\langle 1_a, 2_b|.
\end{aligned} \tag{6}$$

The ASEP with two species is defined by the following equation for the probability of a given configuration $\mathcal{C}$

$$\frac{\mathrm{d}}{\mathrm{d}t} P_t(\mathcal{C}) = \sum_{\mathcal{C}'} M^{0,0,0}(\mathcal{C}, \mathcal{C}') P_t(\mathcal{C}'). \tag{7}$$

where

$$M^{\nu_{10},\nu_{20},\nu_{12}}(\mathcal{C}, \mathcal{C}') = \sum_i \left(E_i^{(10)} + \alpha E_i^{(20)} + \beta E_i^{(12)}\right). \tag{8}$$

Since we are in the presence of several species of particles, we can introduce the *relative* distances covered by particles after an initial time $t = 0$. By this we mean the distance Y^{ij} covered by all the particles of kind i with respect to particles of kind j. It increases of a unity each time a particle of kind i jumps to the right of a particle of kind j, and decreases of a unity when the opposite happens. In our case we have three kind of particles 0, 1, 2, hence we consider Y^{10}, Y^{20}, Y^{12}. The joint probability $P_t(\mathcal{C}, Y^{10}, Y^{20}, Y^{12})$ of being in a configuration $\mathcal{C}$, and having $Y_t^{ij} = Y^{ij}$ satisfies an evolution equation which is better written in terms of a generating function:

$$F_t^{\nu_{10},\nu_{20},\nu_{12}}(\mathcal{C}) = \sum_{Y^{10},Y^{20},Y^{12}} \exp(\nu_{10}Y^{10} + \nu_{20}Y^{20} + \nu_{12}Y^{12}) P_t(\mathcal{C}, Y^{10}, Y^{20}, Y^{12}). \tag{9}$$

The evolution equation satisfied by $F_t^{\nu_{10},\nu_{20},\nu_{12}}(\mathcal{C})$ is

$$\frac{\mathrm{d}}{\mathrm{d}t} F_t^{\nu_{10},\nu_{20},\nu_{12}}(\mathcal{C}) = \sum_{\mathcal{C}'} M^{\nu_{10},\nu_{20},\nu_{12}}(\mathcal{C}, \mathcal{C}') F_t^{\nu_{10},\nu_{20},\nu_{12}}(\mathcal{C}'). \tag{10}$$

One obtains $\langle \exp(\nu_{10}Y_t^{10} + \nu_{20}Y_t^{20} + \nu_{12}Y_t^{12})\rangle$ summing $F_t^{\nu_{10},\nu_{20},\nu_{12}}(\mathcal{C})$ over $\mathcal{C}$; hence, its large time behavior is determined by the largest eigenvalue $\lambda(\nu_{10}, \nu_{20}, \nu_{12})$ of the transition matrix: $M^{\nu_{10},\nu_{20},\nu_{12}}(\mathcal{C}, \mathcal{C}')$

$$\left\langle \exp\left(\nu_{10}Y_t^{10} + \nu_{20}Y_t^{20} + \nu_{12}Y_t^{12}\right)\right\rangle \sim \exp(\lambda(\nu_{10}, \nu_{20}, \nu_{12})t).$$

We find such an eigenvalue by employing the algebraic Bethe ansatz.

Our first step is to find an R-matrix which satisfies the Yang–Baxter equation, the inversion relation and such that its derivative reduces to the linear combination of $E^{(ij)}$'s matrices

$$E^{(10)} + \alpha E^{(20)} + \beta E^{(12)}.$$

Once we have this, we construct the transfer matrix in the usual way as trace of products of L-matrices, which are defined by $L_{a,b}(x, y) = P_{a,b}R_{a,b}(x, y)$ (where $P_{a,b}$ is the permutation operator, which exchanges the component of the tensor product), and we are assured that its logarithmic derivative will be the desired $\sum_i \left(E_i^{(10)} + \alpha E_i^{(20)} + \beta E_i^{(12)}\right)$.

We provide a solution of the Yang–Baxter equation

$$R_{a,b}(y, z)R_{b,c}(x, z)R_{a,b}(x, y) = R_{b,c}(x, y)R_{a,b}(x, z)R_{b,c}(y, z), \tag{11}$$

of the form

$$R(x, y) = 1 + g_{10}(x, y)E^{(10)} + g_{20}(x, y)E^{(20)} + g_{12}(x, y)E^{(12)}, \tag{12}$$

where

$$\begin{aligned} g_{12}(x, y) &= 1 - \frac{1 + \beta(\mathrm{e}^{-y} - 1)}{1 + \beta(\mathrm{e}^{-x} - 1)}; \\ g_{10}(x, y) &= 1 - \mathrm{e}^{x-y}; \\ g_{20}(x, y) &= 1 - \frac{1 + \alpha(\mathrm{e}^{x} - 1)}{1 + \alpha(\mathrm{e}^{y} - 1)}. \end{aligned} \tag{13}$$

We define the monodromy matrix of a system of size N as

$$\mathcal{T}_{a\otimes\mathcal{H}}(x, \vec{\eta}) = L_{a,a_N}(x, \eta_N) \cdots L_{a,a_2}(x, \eta_2)L_{a,a_1}(x, \eta_1). \tag{14}$$

where $L_{a,b}(x, y) = P_{a,b}R_{a,b}(x, y)$, and $P_{a,b}$ is the permutation operator, i.e. $Pv_a \otimes v_b = v_b \otimes v_a$. The transfer matrix is given by

$$T(x, \vec{\eta}) = \mathrm{tr}_a \mathcal{T}_{a\otimes\mathcal{H}}(a, \vec{\eta})$$

Thanks to the Yang–Baxter equation (11) we get

$$\mathcal{T}_{a\otimes\mathcal{H}}(z, \vec{y})\mathcal{T}_{b\otimes\mathcal{H}}(x, \vec{y})R_{a,b}(x, z) = R_{a,b}(x, z)\mathcal{T}_{a\otimes\mathcal{H}}(x, \vec{y})\mathcal{T}_{b\otimes\mathcal{H}}(z, \vec{y}), \tag{15}$$

and, tracing over the auxiliary space, we obtain that the transfer matrices with different values of the spectral parameter commute among themselves

$$[T(x,\vec{\eta}), T(x',\vec{\eta})] = 0. \tag{16}$$

The transition matrix of our system is obtained choosing $\eta_i = 0$, and taking the logarithmic derivative of $T(x,\vec{\eta})$ at $x = 0$

$$M^{\nu_{10},\nu_{20},\nu_{12}}(\mathcal{C},\mathcal{C}') = -T(0,\vec{0})^{-1}\frac{\mathrm{d}T(x,\vec{0})}{dx}\Big|_{x=0}. \tag{17}$$

We come now to the Yang–Baxter algebra, which can be easily read from equation (15). Let us write the monodromy matrix as

$$\mathscr{T}_{a\otimes\mathscr{H}}(x) = \begin{pmatrix} A(x) & B_1(x) & B_2(x) \\ C_1(x) & D_{11}(x) & D_{12}(x) \\ C_2(x) & D_{21}(x) & D_{22}(x) \end{pmatrix}. \tag{18}$$

The transfer matrix can then be written as

$$T(x) = A(x) + D_{11}(x) + D_{22}(x), \tag{19}$$

where we have simplified the notation omitting $\vec{\zeta}$ which is fixed to be zero. In order to make manifest the nested structure of the R-matrix, let us rewrite it as

$$R(x,y) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & (1-\mathrm{e}^{x-y})\,\mathrm{e}^{\nu_{10}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \left(1-\frac{1+\alpha(\mathrm{e}^{x}-1)}{1+\alpha(\mathrm{e}^{y}-1)}\right)\mathrm{e}^{\nu_{20}} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{x-y} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & R^{(1)}_{11,11} & R^{(1)}_{11,12} & 0 & R^{(1)}_{11,21} & R^{(1)}_{11,22} \\ 0 & 0 & 0 & 0 & R^{(1)}_{12,11} & R^{(1)}_{12,12} & 0 & R^{(1)}_{12,21} & R^{(1)}_{12,22} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+\alpha(\mathrm{e}^{x}-1)}{1+\alpha(\mathrm{e}^{y}-1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & R^{(1)}_{21,11} & R^{(1)}_{21,12} & 0 & R^{(1)}_{21,21} & R^{(1)}_{21,22} \\ 0 & 0 & 0 & 0 & R^{(1)}_{22,11} & R^{(1)}_{22,12} & 0 & R^{(1)}_{22,21} & R^{(1)}_{22,22} \end{pmatrix} \tag{20}$$

Where the matrix elements $R^{(1)}_{ij,lk}$, which actually depend on x and y, are gathered in a matrix, which is given by

$$R^{(1)}(x,y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1+\beta(\mathrm{e}^{-y}-1)}{1+\beta(\mathrm{e}^{-x}-1)} & 0 & 0 \\ 0 & \left(1-\frac{1+\beta(\mathrm{e}^{-y}-1)}{1+\beta(\mathrm{e}^{-x}-1)}\right)\mathrm{e}^{\nu_{12}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{21}$$

This is nothing else than the R-matrix corresponding to the ASEP with a single species presented in equation (A.3) in appendix A, with a different parameterization

$$\mathrm{e}^{y_i} \to \frac{1}{1+\beta(\mathrm{e}^{-y_i}-1)}. \tag{22}$$

With this notation we can write the commutation rules of the operators A, B_i, C_i and D_{ij} appearing in $\mathscr{T}$, for different values of the spectral parameters

$$[A(x), A(y)] = 0; \tag{23}$$

$$A(x)B_1(y) = \frac{e^x}{e^{\nu_{10}}(e^x - e^y)}B_1(y)A(x) + \frac{e^y}{e^{\nu_{10}}(e^y - e^x)}B_1(x)A(y); \tag{24}$$

$$A(x)B_2(y) = \frac{1+\alpha(e^x - 1)}{e^{\nu_{20}}\alpha(e^x - e^y)}B_2(y)A(x) + \frac{1+\alpha(e^y - 1)}{e^{\nu_{20}}\alpha(e^y - e^x)}B_2(x)A(y); \tag{25}$$

$$B_i(x)B_j(y) = B_l(y)B_k(x)R^{(1)}_{ij,lk}(x,y); \tag{26}$$

$$D_{1j}(x)B_k(y) = \frac{e^y}{e^{\nu_{10}}(e^y - e^x)}\big(B_m(y)D_{1n}(x)R^{(1)}_{jk,mn}(x,y) - B_j(x)D_{1k}(y)\big); \tag{27}$$

$$D_{2j}(x)B_k(y) = \frac{1+\alpha(e^y - 1)}{e^{\nu_{20}}\alpha(e^y - e^x)}(B_m(y)D_{2n}(x)R^{(1)}{}_{jk,mn}(x,y) - B_j(x)D_{2k}(y)). \tag{28}$$

The ansatz for an eigenvector of the transfer matrix, taking into account the non-commutativity of the B_i s for different i, is given by

$$|\Psi^{M_1,M_2}(y_1,\ldots,y_r)\rangle = \sum_{i_1,\ldots,i_r}\Psi^{M_1,M_2}_{i_1,\ldots,i_r}B_{i_1}(y_1)\cdots B_{i_r}(y_r)\|1\rangle, \tag{29}$$

where $\|1\rangle$ is the reference state, defined by

$$\|1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}\otimes\cdots\otimes\begin{pmatrix}1\\0\\0\end{pmatrix},$$

and it is an eigenstate separately of $A(x)$, $D_{11}(x)$ and $D_{22}(x)$

$$A(x)\|1\rangle = \|1\rangle, \qquad D_{11}(x)\|1\rangle = e^{N\nu_{10}}(1-e^x)^N\|1\rangle,$$
$$D_{22}(x)\|1\rangle = (\alpha\, e^{\nu_{20}})^N(1-e^x)^N\|1\rangle,$$

and correspond to a completely empty system. The labels M_1, M_2 in equation (29) mean that we require $M_1 B$ s of type 1 and $M_2 B$ s of type 2, i.e. we are restricting to the sector with M_1 particles of type 1 and M_2 particles of type 2.

The eigenvector equation for $|\Psi^{M_1,M_2}(y_1,\ldots,y_r)\rangle$ puts constraints on the y_i s. Let us first apply the operator $A(x)$ to $|\Psi^{M_1,M_2}(y_1,\ldots,y_r)\rangle$. We get a wanted term, i.e. a term proportional to the vector we start from, of the form

$$\left(\frac{e^x}{e^{\nu_{10}}}\right)^{M_1}\left(\frac{1+\alpha(e^x-1)}{e^{\nu_{20}}\alpha}\right)^{M_2}\prod_{i=1}^{M_1+M_2}\frac{1}{(e^x - e^{y_i})}|\Psi^{M_1,M_2}(y_1,\ldots,y_r)\rangle \tag{30}$$

and unwanted terms of the form

$$\frac{1}{(e^{y_j}-e^x)}\left(\frac{e^{y_j}}{e^{\nu_{10}}}\right)^{M_1}\left(\frac{1+\alpha(e^{y_j}-1)}{e^{\nu_{20}}\alpha}\right)^{M_2}\prod_{i\neq j}^{M_1+M_2}\frac{1}{(e^{y_j}-e^{y_i})}B(x)\otimes B(y_{j+1})\otimes\cdots\otimes B(y_{j-1})$$
$$\times M(y_j,\vec{y})M(y_{j-1},\vec{y})\cdots M(y_1,\vec{y})\Psi^{M_1,M_2}(y_1,\ldots,y_r)\|1\rangle \tag{31}$$

with

$$M(y_j,\vec{y}) = R^{(1)}_{i_1i_2,j_1j_2'}(y_j,y_{j+1})R^{(1)}_{j_2'i_3,j_2j_3'}(y_j,y_{j+2})\cdots R^{(1)}_{j_{r-1}'i_r,j_{r-1},j_r}(y_j,y_{j-1}).$$

The unwanted terms have to cancel with similar terms coming from the action of $D_{11}(x,\vec{y}) + D_{22}(x,\vec{y})$. From the action of D_{kk} we get a wanted term

$$\omega_k(\vec{y})(1-e^x)^N\prod_{i=1}^{M_1+M_2}\frac{1}{(e^{y_i}-e^x)}B(y_1)\otimes\cdots\otimes B(y_{M_1+M_2})T^{(1)}_{kk}(x,\vec{y})\Psi^{M_1,M_2}(y_1,\ldots,y_r)\|1\rangle, \tag{32}$$

with

$$\omega_1(\vec{y}) = \mathrm{e}^{N\nu_{10}} \prod_{i=1}^{M_1+M_2} \left(\frac{\mathrm{e}^{y_i}}{\mathrm{e}^{\nu_{10}}}\right), \qquad \omega_2(\vec{y}) = \mathrm{e}^{N\nu_{20}}\alpha^N \prod_{i=1}^{M_1+M_2} \left(\frac{1+\alpha(\mathrm{e}^{y_i}-1)}{\mathrm{e}^{\nu_{20}}\alpha}\right),$$

and

$$T^{(1)}(x,\vec{y}) = L^{(1)}_{a,a_{M_1+M_2}}(x, y_{M_1+M_2})\cdots L^{(1)}_{a,a_1}(x,y_1) \tag{33}$$

is just the monodromy matrix of TASEP with a single species as explained in appendix A. We also get an unwanted term

$$\omega_k(\vec{y})\frac{(1-\mathrm{e}^{y_j})^N}{\mathrm{e}^x-\mathrm{e}^{y_j}} \prod_{i\neq j}^{M_1+M_2} \frac{1}{(\mathrm{e}^{y_i}-\mathrm{e}^{y_j})} B(x)\otimes B(y_{j+1})\otimes\cdots\otimes B(y_{j-1})$$

$$M(y_j,\vec{y})M(y_{j-1},\vec{y})\cdots M(y_1,\vec{y})\Psi^{M_1,M_2}(y_1,\ldots,y_r)T^{(1)}_{kk}(y_j,\vec{y})\Psi^{M_1,M_2}(y_1,\ldots,y_r)\|1\rangle.$$

In order to get the cancellation of the unwanted terms we first have to diagonalize $\omega_1(\vec{y})T^{(1)}_{11} + \omega_2(\vec{y})T^{(1)}_{22}$. This is the transfer matrix of a TASEP with a single species and twisted boundary condition and can be diagonalized by the algebraic Bethe ansatz as in the case of non-twisted boundary conditions, as done in [20], we briefly recall how it works in appendix A. Here we simply apply the results explained there. One has only to be careful in translating the parameters $\mathrm{e}^{\tilde{y}_i}$ appearing in the appendix, following equation (22)

$$\mathrm{e}^{\tilde{y}_i} = \frac{1}{1+\beta(\mathrm{e}^{-y_i}-1)} = \frac{Y_i-1}{bY_i-1}, \tag{34}$$

where we have defined $Y_i = 1-\mathrm{e}^{y_i}$, $b = 1-\beta$, and remember that we consider only the sector with M_2 particles in a system of size $\tilde{N} = M_1+M_2$.

For the auxiliary spectral parameters Zs we get the Bethe equation (A.14) which now reads

$$\left(\frac{\mathrm{e}^{\nu_{20}}\alpha}{\mathrm{e}^{\nu_{10}}}\right)^N \prod_{i=1}^{M_1+M_2} \frac{(1-\alpha Y_i)(bY_i-1)}{(1-Y_i)(bY_i-1-Z_j(Y_i-1))} \prod_{k\neq j}^{M_2}\left(-\frac{Z_j}{Z_k}\right) = \left(\alpha\frac{\mathrm{e}^{\nu_{12}}\,\mathrm{e}^{\nu_{20}}}{\mathrm{e}^{\nu_{10}}}\right)^{M_1+M_2}. \tag{35}$$

While the cancellation of the unwanted terms coming from A and D_{kk} leads to a second Bethe equation

$$\frac{(1-Y_j)^{M_1}(1-\alpha Y_j)^{M_2}(bY_j-1)^{M_2}}{\mathrm{e}^{N\nu_{10}}Y_j^N\prod_{i=1}^{M_1+M_2}(1-Y_i)} = \left(\frac{\alpha\,\mathrm{e}^{\nu_{12}}\,\mathrm{e}^{\nu_{20}}}{\mathrm{e}^{\nu_{10}}}\right)^{M_2}\prod_{k=1}^{M_2}(bY_j-1-Z_k(Y_j-1)). \tag{36}$$

The eigenvalue of the transfer matrix can be read from equations (30, 32) and equation (A.15)

$$\Lambda(x) = \left(\frac{\mathrm{e}^x}{\mathrm{e}^{\nu_{10}}}\right)^{M_1}\left(\frac{1+\alpha(\mathrm{e}^x-1)}{\mathrm{e}^{\nu_{20}}\alpha}\right)^{M_2}\prod_{i=1}^{M_1+M_2}\frac{1}{(\mathrm{e}^x-\mathrm{e}^{y_i})}$$
$$+(1-\mathrm{e}^x)^N\,\mathrm{e}^{\nu_{12}M_2}\omega_1(\vec{y})\prod_{i=1}^{r}\left(1-\mathrm{e}^xZ_i\right)\prod_{i=1}^{M_1+M_2}\frac{1}{(\mathrm{e}^x-\mathrm{e}^{y_i})}. \tag{37}$$

Taking the logarithmic derivative in $x = 0$ we get the eigenvalue of the transition matrix λ

$$\lambda = -\frac{1}{\Lambda(0)}\frac{\mathrm{d}\Lambda(x)}{\mathrm{d}x}\bigg|_0 = -M_1-\alpha M_2+\sum_{i=1}^{M_1+M_2}\frac{1}{Y_i}. \tag{38}$$

3. Analysis of the Bethe equations

For convenience of notation, we divide the Ys into two sets, $Y_i^{(1)}$ with $i = 1, \ldots, M_1$, and $Y_i^{(\alpha)}$ with $i = 1, \ldots, M_2$. The solution of the Bethe equations wanted behaves in the limit $\nu_{ij} \to 0$ as $Y_i^{(1)} \to 1$, $Y_i^{(\alpha)} \to 1/\alpha$ and $Z_k \to Z_k^{(0)}$, where $Z_k^{(0)}$ have to be determined. Actually we will see that $Z_k^{(0)}$ s depend on how the limit is taken. For this reason, we will redefine $\nu_{ij} \to \nu\nu_{ij}$ and take the limit $\nu \to 0$ keeping ν_{ij} fixed. What happens is that the $Z_k^{(0)}$ s depend on these ν_{ij} (or more precisely on their ratios).

From equations (35) and (36) we get $\left(e^{\nu_{20}M_2+\nu_{10}M_1}\alpha^{M_2}\prod_{i=1}^{M_1}Y_i^{(1)}\prod_{i=1}^{M_2}Y_i^{(\alpha)}\right)^N = 1$ and by continuity

$$e^{\nu_{20}M_2+\nu_{10}M_1}\alpha^{M_2}\prod_{i=1}^{M_1}Y_i^{(1)}\prod_{i=1}^{M_2}Y_i^{(\alpha)} = 1 \tag{39}$$

Let us introduce the following auxiliary variables

$$\begin{aligned} C &= e^{(\nu_{12}+\nu_{20}-\nu_{10})M_2+\nu_{10}N}\prod_{i=1}^{M_1+M_2}(1-Y_i),\\ K &= -\frac{e^{\nu_{12}(M_1+M_2)}}{(\alpha\, e^{\nu_{20}-\nu_{10}})^{N-M_1-M_2}}\frac{\prod_{i=1}^{M_1+M_2}(1-Y_i)\prod_{i=1}^{M_2}Z_k}{\prod_{i=1}^{M_1+M_2}(1-\alpha Y_i)(bY_i-1)}. \end{aligned}\tag{40}$$

The Bethe equations become

$$(-Z_j)^{M_2} = K\prod_{i=1}^{M_1+M_2}(bY_i - 1 - Z_j(Y_i-1)) \tag{41}$$

$$(1-Y_j)^{M_1}(1-\alpha Y_j)^{M_2}(bY_j-1)^{M_2} = C\alpha^{M_2}Y_j^N\prod_{k=1}^{M_2}(bY_j-1-Z_k(Y_j-1)). \tag{42}$$

We note that if we keep K as an unknown, combining equation (39) with equations (41) and (42) we recover the definition of K given in equation (40). Hence from now on our basic equations are (39), (41) and (42). Following steps similar to those in [10] we obtain the following representation for the eigenvalue, which generalizes equations (33) (34) and (36) of [10]

$$\lambda = -\sum_{n=1}^{\infty}\frac{C^n}{n}\left[\oint_1+\oint_{1/\alpha}\right]\frac{dy}{2\pi i}\frac{1}{y^2}[Q(y)]^n \tag{43}$$

where for us

$$Q(y) = \frac{y^N\alpha^{M_2}\prod_{k=1}^{M_2}(by-1-Z_k(y-1))}{(1-\alpha y)^{M_2}(by-1)^{M_2}(1-y)^{M_1}}. \tag{44}$$

Taking the logarithm of equation (39) we get

$$\nu_{10}M_1+\nu_{20}M_2 = -\sum_{n=1}^{\infty}\frac{C^n}{n}\left[\oint_1+\oint_{1/\alpha}\right]\frac{dy}{2\pi i}\frac{1}{y}[Q(y)]^n, \tag{45}$$

while equation (41) becomes (after having taken the logarithm)

$$\begin{aligned} M_2\log(-Z_j) &= \log K + 2i\pi j + M_1\log(b-1) + M_2\log(b/\alpha - 1 - Z_j(1/\alpha-1))\\ &\quad+\sum_{n=1}^{\infty}\frac{C^n}{n}\left[\oint_1+\oint_{1/\alpha}\right]\frac{dy}{2\pi i}\frac{b-Z_j}{by-1-Z_j(y-1)}Q(y)^n. \end{aligned}\tag{46}$$

Taking the logarithm of the first equation in (40) we get also the equation

$$\nu_{10}(N-M_2)+(\nu_{20}+\nu_{12})M_2=-\sum_{n=1}^{\infty}\frac{C^n}{n}\left[\oint_1+\oint_{1/\alpha}\right]\frac{\mathrm{d}y}{2\pi\mathrm{i}}\frac{1}{y-1}[Q(y)]^n. \tag{47}$$

Now, if we redefine $\nu_{ij}\to\nu\nu_{ij}$, we see that equations (37), (45)–(47) give in an implicit form the power expansion in ν of λ. What one should do in principle is to expand $\log(K)$ and Z_j in powers of C, use equation (46) and a combination of equations (45) and (47) to derive the nth-order term of these expansions in terms of lower orders terms, and then work out the expansion of λ in powers of ν. This way one would find the cumulants of the total number of particles flown $\nu_{10}Y^{(10)}+\nu_{20}Y^{(20)}+\nu_{12}Y^{(12)}$

$$\lambda(\nu)=\lim_{t\to\infty}\frac{\log\left\langle \mathrm{e}^{\nu(\nu_{10}Y^{(10)}+\nu_{20}Y^{(20)}+\nu_{12}Y^{(12)})}\right\rangle}{t}=\sum_n\frac{\langle(\nu_{10}Y^{(10)}+\nu_{20}Y^{(20)}+\nu_{12}Y^{(12)})^n\rangle_c}{t}\nu^n. \tag{48}$$

Concretely this is of course quite laborious and one does not find any illuminating formulae in general, but one can easily find at least the velocities. Let us work out explicitly the average of the total velocity of the particles of second type. For this we have to chose $\nu_{10}=0$, $\nu_{20}=-\nu_{12}=1$, and the velocity is given simply by the linear term of the expansion of λ in terms on ν. In the limit $\nu\to 0$, the Z_j s satisfy a very simple equation

$$\left(Z_j^{(0)}\right)^{M_2}=(-1)^{(M_2)}K^{(0)}(b-1)^{M_1}\left[\frac{b}{\alpha}-1-Z_j^{(0)}\left(\frac{1}{\alpha}-1\right)\right]^{M_2}, \tag{49}$$

whose solution is

$$Z_j^{(0)}=\frac{(\alpha-b)\,\mathrm{e}^{\frac{2\pi ij}{M_2}}[K^{(0)}(b-1)^{M_1}]^{1/M_2}}{\alpha-(1-\alpha)\,\mathrm{e}^{\frac{2\pi ij}{M_2}}[K^{(0)}(b-1)^{M_1}]^{1/M_2}}. \tag{50}$$

$Z_j^{(0)}$ are now expressed in terms of a single unknown $K^{(0)}$ which is determined taking the first order in C of the constraint equation (47)

$$\left[\oint_1+\oint_{1/\alpha}\right]\frac{\mathrm{d}y}{2\pi\mathrm{i}(y-1)}Q^{(0)}(y)=0, \tag{51}$$

where $Q^{(0)}$ is the value of Q for $\nu=0$, which is given by

$$Q^{(0)}(y)=\frac{y^N}{(1-y)^{M_1}}\left(\frac{\alpha^{M_2}}{(1-\alpha y)^{M_2}}-K^{(0)}\frac{(b-1)^{M_1+M_2}}{(by-1)^{M_2}}\right). \tag{52}$$

Then for $K^{(0)}$ we find

$$K^{(0)}=\frac{\left[\oint_1+\oint_{1/\alpha}\right]\frac{\mathrm{d}y}{2\pi\mathrm{i}}\frac{y^N\alpha^{M_2}}{(1-y)^{M_1+1}(1-\alpha y)^{M_2}}}{\left[\oint_1+\oint_{1/\alpha}\right]\frac{\mathrm{d}y}{2\pi\mathrm{i}}\frac{y^N(b-1)^{M_1+M_2}}{(1-y)^{M_1+1}(by-1)^{M_2}}}. \tag{53}$$

Note that, as stated before, the value of $K^{(0)}$, and hence of the $\zeta_k^{(0)}$ s, depends on the choice of ν_{ij}. Had we chosen different values for ν_{10}, ν_{20} and ν_{12}, we would have found a different value for $K^{(0)}$.

Now we have all the ingredient we need to derive the velocity of the particles of kind 2. We have simply to consider the linear part of equations (43) and (45)

$$v_2=\lim_{\nu\to 0}\frac{\lambda(\nu)}{\nu}=M_2\frac{\left[\oint_1+\oint_{1/\alpha}\right]\frac{\mathrm{d}y}{2\pi\mathrm{i}}\frac{Q^{(0)}(y)}{y^2}}{\left[\oint_1+\oint_{1/\alpha}\right]\frac{\mathrm{d}y}{2\pi\mathrm{i}}\frac{Q^{(0)}(y)}{y}}, \tag{54}$$

which is more conveniently written in terms of the following two auxiliary functions

$$F^{\alpha}_{N,M_1,M_2} = \left[\oint_1 + \oint_{1/\alpha}\right] \frac{\mathrm{d}y}{2\pi} \frac{y^N}{(y-1)^{M_1}(\alpha y - 1)^{M_2}}, \tag{55}$$

$$F^{b}_{N,M_1,M_2} = \left[\oint_1 + \oint_{1/\alpha}\right] \frac{\mathrm{d}y}{2\pi} \frac{y^N}{(y-1)^{M_1}(1 - by)^{M_2}} \tag{56}$$

(note that the contours of integration are the same for the two integrals)

$$v_2 = M_2 \frac{F^{\alpha}_{N-2,M_1,M_2} F^{b}_{N,M_1+1,M_2} - F^{\alpha}_{N,M_1+1,M_2} F^{b}_{N-2,M_1,M_2}}{F^{\alpha}_{N-1,M_1,M_2} F^{b}_{N,M_1+1,M_2} - F^{\alpha}_{N,M_1+1,M_2} F^{b}_{N-1,M_1,M_2}}. \tag{57}$$

A check of our formula comes setting $\alpha = 1$ and $b = 0$. In such a case we can compute both F^{α} and F^{b} exactly[1]

$$F^{\alpha=1}_{N,M_1,M_2} = \binom{N}{M_1 + M_2 - 1}, \qquad F^{b=0}_{N,M_1,M_2} = \binom{N}{M_1 - 1}, \tag{58}$$

and for the velocity we get

$$v_2 = M_2 \frac{N - 2M_1 - M_2}{N-1}. \tag{59}$$

This result has a quite simple explanation in terms of ASEP with a single species. Indeed, when $\alpha = 1$ and $b = 0$ we can identify particles of species 1 and species 2, then the total velocity of the particles is well known to be [9]

$$v_1 + v_2 = \frac{(M_1 + M_2)(N - M_1 - M_2)}{N-1},$$

on the other hand we could identify particles of species 2 and holes, and deduce easily the total velocity of particles of type 1

$$v_1 = \frac{M_1(N - M_1)}{N-1}.$$

Taking the difference of the two formulae above, we recover the expression in equation (59).

4. Large-N limit of the velocity

We want now to consider the limit $N \to \infty$ with nonzero densities of particles of both species $\rho_1 = M_1/N$ and $\rho_2 = M_2/N$. To find the asymptotic formula for the velocity we have simply to determine the asymptotic of F^{α} and F^{b}, which are easily given by the steepest descent method. Both integrals have two saddle points which correspond to the solutions of the equation

$$\frac{1}{y} - \frac{\rho_1}{y-1} - \frac{\rho_2 \kappa}{\kappa y - 1} = 0 \tag{60}$$

where $\kappa = \alpha$ for F^{α}, and $\kappa = b$ while for F^{b}. The expression for the saddle points is

$$y^{\pm}_{\kappa} = \frac{\kappa + 1 - \rho_1 - \kappa\rho_2 \pm \sqrt{(\kappa + 1 - \rho_1 - \kappa\rho_2)^2 - 4\kappa(1 - \rho_1 - \rho_2)}}{2\kappa(1 - \rho_1 - \rho_2)}. \tag{61}$$

It is easy to realize that both saddle points are on the real line, one is situated between 1 and $1/\kappa$, the other is situated to the right of $1/\kappa$.

[1] As explained in [10] when $\alpha = 1$ one has to take a single contour integral around 1.

When we compute F^α we can merge the contour around 1 with the one around $1/\alpha$ and let the resulting contour pass through y_α^+, hence F^α is dominated by the contribution from y_α^+. In computing F^b the integral around $1/\alpha$ gives no contribution because the integrand is holomorphic there. The contour integral around 1 can now be deformed only to pass through y_b^-, because of the singularity present in $1/b$; hence, F^b is dominated by the contribution from y_b^-.

In conclusion we get, for generic α and b,

$$v_2 = M_2\left(\frac{1}{y_\alpha^+} + \frac{1}{y_b^-} - 1\right). \tag{62}$$

By the same procedure, one can find the total velocity of the particles of kind 1

$$v_1 = \frac{N}{y_\alpha^+ y_b^-} - (N - M_1)\left(\frac{1}{y_\alpha^+} + \frac{1}{y_b^-} - 1\right). \tag{63}$$

From equation (61) we see that the non-analyticities of the total velocities (62) and (63) are located at the zeros of the square root, i.e. at

$$(\alpha + 1 - \rho_1 - \alpha\rho_2)^2 - 4\alpha(1 - \rho_1 - \rho_2) = 0 \tag{64}$$

and at

$$(b + 1 - \rho_1 - b\rho_2)^2 - 4b(1 - \rho_1 - \rho_2) = 0. \tag{65}$$

Equation (64) has solutions:

- for $\alpha < 1$: $\rho_2 = 0$, $\rho_1 = 1 - \alpha$;
- for $\alpha > 1$: $\rho_1 = 0$, $\rho_2 = \frac{\alpha-1}{\alpha}$.

Equation (65) has a solution only for $\rho_2 = 0$ and $\rho_1 = 1 - b$. This means that as long as the densities of the particles are nonzero, the velocity of the particles is analytic in all its parameters α, β, ρ_1 and ρ_2. This result shows that the behavior of the system in the presence of nonzero densities of particle of each species is quite different from the case dealt with in [10, 12] where they consider a single particle of type 2, which in the large size limit corresponds to $\rho_2 = 0$. In that case the velocity is non-analytic in α or β at

$$\alpha = 1 - \rho_1 \qquad \text{and} \qquad \beta = \rho_1. \tag{66}$$

These non-analyticities are present whenever $\rho_2 = 0$, i.e. whenever we consider a fixed number of particles of type 2 and let the size of the system go to infinity.

5. Conclusions

In this paper, we have studied an ASEP with two species of particles and different hopping rates. We have formulated the computation of the cumulants of the currents as an eigenvalue equation, and we have shown that this leads to an integrable (in the manner of Yang–Baxter) transition matrix. This has allowed us to employ the formalism of the algebraic Bethe ansatz to solve the problem, by finding the Bethe equations for an arbitrary number of particles of each species. The analysis of the Bethe equations gives in principle all the cumulants of the currents. We found the exact formula for the velocity of the particles of type 2, and computed its limit when the size of the system goes to infinity, keeping nonzero densities for the particles. We find this way that, when the densities are different from *zero*, the total velocity of each species of particles is analytic in all the parameters. In order to understand better the features of the system in such a situation one should resort to the matrix product ansatz for determining for example the density profile of the particles.

Our work can be extended in different directions. First, we think it would be interesting to use the Bethe equation we found, to compute the spectral gap as a function of the hopping parameters α and β. We have briefly discussed in the appendices the extension of the problem to a larger number of species and different hopping rates; it would also be nice to work out the average velocity of particles of a given type as functions of the hopping parameters. Another interesting possibility is to consider the problem on a lattice with open ends and letting particles flow in and out of the system. We plan to come back to these issues soon.

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Appendix A. Algebraic Bethe ansatz for the ASEP with one species

Let us consider the ASEP consisting of m particles and r holes on a ring of size $\tilde{N} = p + r$. Each particle can jump into a neighbor site only if the site is empty. The probability for jumping forward is $q \cdot \mathrm{d}t$, while the probability for jumping backward is $p \cdot \mathrm{d}t$. Following [9] we consider the total distance covered by all the particles in a time t, denoted by Y_t. In order to determine the behavior of Y_t we look at the joint probability $P_t(\mathcal{C}, Y)$ of being at time t in a configuration $\mathcal{C}$ and having all the particles covered a total distance $Y_t = Y$. The generating function

$$F_t(\mathcal{C}) = \sum_{Y=0}^{\infty} \mathrm{e}^{\nu_{12} Y} P_t(\mathcal{C}, Y),$$

which satisfies the following evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t} F_t(\mathcal{C}) = \sum_{\mathcal{C}'} [M_0(\mathcal{C}, \mathcal{C}') + \mathrm{e}^{\nu_{12}} M_1(\mathcal{C}, \mathcal{C}') + \mathrm{e}^{-\nu_{12}} M_{-1}(\mathcal{C}, \mathcal{C}')] F_t(\mathcal{C}'), \tag{A.1}$$

where $M_1(\mathcal{C}, \mathcal{C}')\,\mathrm{d}t$ is the transition probability for going from the configuration $\mathcal{C}'$ to the configuration $\mathcal{C}$ *and* moving a particle *forward* of one step, while $M_{-1}(\mathcal{C}, \mathcal{C}')\,\mathrm{d}t$ correspond to a particle moving *backward* of one step and M_0 is the diagonal part. The large time behavior of $\langle \mathrm{e}^{\nu_{12} Y_t} \rangle$ is determined by the largest eigenvalue $\lambda(\nu_{12})$ of the matrix transition matrix $M_0(\mathcal{C}, \mathcal{C}') + \mathrm{e}^{\nu_{12}} M_1(\mathcal{C}, \mathcal{C}') + \mathrm{e}^{-\nu_{12}} M_{-1}(\mathcal{C}, \mathcal{C}')$.

The transition matrix $M_0(\mathcal{C}, \mathcal{C}') + \mathrm{e}^{\nu_{12}} M_1(\mathcal{C}, \mathcal{C}') + \mathrm{e}^{-\nu_{12}} M_{-1}(\mathcal{C}, \mathcal{C}')$ can be diagonalized by means of the algebraic Bethe ansatz [19, 20]. In our case, we are led to consider

$$R_{a,b}(x, y) = 1 + \lambda(x, y) E \tag{A.2}$$

where

$$\lambda(x, y) = \frac{\mathrm{e}^{\frac{x-y}{2}} - \mathrm{e}^{\frac{y-x}{2}}}{p\,\mathrm{e}^{\frac{x-y}{2}} - q\,\mathrm{e}^{\frac{y-x}{2}}}.$$

The matrix $R_{a,b}(x, y)$ acts on $V_a \otimes V_b \to V_a \otimes V_b$, where $V = \mathbb{C}^2$, and in the basis $(|0\rangle, |1\rangle)_a \otimes (|0\rangle, |1\rangle)_b$ the matrix E reads

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & p\,\mathrm{e}^{-\nu_{12}} & 0 \\ 0 & q\,\mathrm{e}^{\nu_{12}} & -p & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.3}$$

We define the matrices $L_{a,a_i}(x, y_i) = P_{a,a_i} R_{a,a_i}(x, y_i)$, where $P_{a,b}$ is the permutation operator. We introduce also a matrix Ω which acts only on the auxiliary space, in a diagonal way

$$\Omega(\vec{y}) = \begin{pmatrix} \omega_1(\vec{y}) & 0 \\ 0 & \omega_2(\vec{y}) \end{pmatrix}$$

and its entries can depend on some auxiliary parameters $\vec{y}$. The reason for considering such a generalization of the problem we started from, which in facts correspond to Ω equal to the identity, comes from the ASEP with two species, as seen in the text. For a system of size $\tilde{N}$ the monodromy matrix is constructed by means of $L_{a,a_i}(x, \tilde{y}_i)$ [2]

$$\mathscr{T}_{a\otimes\mathscr{H}}(x, \vec{\tilde{y}}) = L_{a,a_{\tilde{N}}}(x, \tilde{y}_{\tilde{N}}) \cdots L_{a,a_1}(x, \tilde{y}_1).$$

The transfer matrix is given by

$$T(x, \vec{y}, \vec{\tilde{y}}) = \mathrm{tr}_a[\Omega(\vec{y})\mathscr{T}_{a\otimes\mathscr{H}}(\vec{\tilde{y}})].$$

The fact that $[\Omega(\vec{y}) \otimes \Omega(\vec{y}), R(x, x')] = 0$, combined with the Yang–Baxter equation, implies that the transfer matrices with different parameters x and x' commute among themselves. The transition matrix of the ASEP is obtained as the logarithmic derivative of the transfer matrix in zero (at $\Omega = \mathrm{Id}$ and $y_i = 0$).

Let us write the monodromy matrix in the auxiliary space as

$$\mathscr{T}(x, \vec{\tilde{y}}) = \begin{pmatrix} A(x, \vec{\tilde{y}}) & B(x, \vec{\tilde{y}}) \\ C(x, \vec{\tilde{y}}) & D(x, \vec{\tilde{y}}) \end{pmatrix}, \tag{A.4}$$

the algebraic Bethe ansatz proceeds by constructing an eigenvector acting with $B(\zeta_i, \vec{\tilde{y}})$ on a reference state. Our reference state is $|1\rangle = \binom{1}{0} \otimes \cdots \otimes \binom{1}{0}$ corresponding to the completely full system, which is an eigenvector of the transfer matrix. Indeed we note that $A(x, \vec{\tilde{y}})|1\rangle = |1\rangle$, $D(x, \vec{\tilde{y}})|1\rangle = \prod_{k=1}^{\tilde{N}} \left(\frac{p\lambda(x,\tilde{y}_k)}{\mathrm{e}^{\nu_{12}}}\right)|1\rangle$, $C(x, \vec{\tilde{y}})|1\rangle = 0$. Hence the eigenvalue of$|1\rangle$ is $\omega_1(\vec{y})A(x, \vec{\tilde{y}}) + \omega_2(\vec{y})D(x, \vec{\tilde{y}}) = \omega_1(\vec{y}) + \omega_2(\vec{y}) \prod_{k=1}^{\tilde{N}} (p\lambda(x, \tilde{y}_k))$. We search now for eigenvectors of the form

$$|(\zeta_1, \ldots, \zeta_r)\rangle = B(\zeta_1) \cdots B(\zeta_r)|1\rangle \tag{A.5}$$

and use the Yang–Baxter algebra, satisfied by the operators $A(x), B(x), C(x), D(x)$ as a consequence of the Yang–Baxter equation:

$$[A(x), A(y)] = 0, \tag{A.6}$$

$$A(x)B(z) = \frac{\mathrm{e}^{\nu_{12}}}{p\lambda(z, x)} B(z)A(x) - \frac{\mathrm{e}^{\nu_{12}}(1 - p\lambda(z, x))}{p\lambda(z, x)} B(x)A(z) \tag{A.7}$$

$$B(z)B(x) = B(x)B(z) \tag{A.8}$$

$$D(x)B(z) = \frac{\mathrm{e}^{\nu_{12}}}{p\lambda(x, z)} B(z)D(x) - \frac{\mathrm{e}^{\nu_{12}}(1 - q\lambda(x, z))}{p\lambda(x, z)} B(x)D(z). \tag{A.9}$$

[2] Note that ys and $\tilde{y}$ can and will be in general different quantities.

The requirement $|(\zeta_1,\ldots,\zeta_r)\rangle$ to be an eigenvector can be expressed in terms of a Bethe equation

$$\frac{\omega_2(\vec{y})}{\omega_1(\vec{y})}\prod_{i\neq j}\left(\frac{\lambda(\zeta_i,\zeta_j)}{\lambda(\zeta_j,\zeta_i)}\right)\prod_{k=1}^{\tilde{N}}(p\lambda(\zeta_j,\tilde{y}_k)) = \mathrm{e}^{\nu_{12}\tilde{N}}, \tag{A.10}$$

which fixes the values of $\vec{\zeta}$. The eigenvalue of the transfer matrix is given by

$$\Lambda(x) = \omega_1(\vec{y})\prod_{i=1}^{r}\left(\frac{\mathrm{e}^{\nu_{12}}}{p\lambda(\zeta_i,x)}\right) + \omega_2(\vec{y})\prod_{i=1}^{r}\left(\frac{\mathrm{e}^{\nu_{12}}}{p\lambda(x,\zeta_i)}\right)\prod_{k=1}^{\tilde{N}}\left(\frac{p\lambda(x,\tilde{y}_k)}{\mathrm{e}^{\nu_{12}}}\right). \tag{A.11}$$

We consider now the limit $p\to 0, q=1$. In order to do that without getting a singular limit, we have to change the spectral parameters into $\zeta_i \to \zeta_i - \frac{\log(p)}{2}$. Then the Bethe equations turn into the form

$$\frac{\omega_2(\vec{y})}{\omega_1(\vec{y})}\prod_{i\neq j}\left(-\frac{p\,\mathrm{e}^{(\zeta_j-\zeta_i)/2} - q\,\mathrm{e}^{(\zeta_i-\zeta_j)/2}}{p\,\mathrm{e}^{(\zeta_i-\zeta_j)/2} - q\,\mathrm{e}^{(\zeta_j-\zeta_i)/2}}\right)\prod_{k=1}^{\tilde{N}}\left(\frac{\mathrm{e}^{(\zeta_j-\tilde{y}_k)/2} - p\,\mathrm{e}^{(\tilde{y}_k-\zeta_j)/2}}{\mathrm{e}^{(\zeta_j-\tilde{y}_k)/2} - q\,\mathrm{e}^{(\tilde{y}_k-\zeta_j)/2}}\right) = \mathrm{e}^{\nu_{12}\tilde{N}} \tag{A.12}$$

and for $p\to 0$ and $q=1$ we get

$$\frac{\omega_2(\vec{y})}{\omega_1(\vec{y})}\prod_{i\neq j}(-\mathrm{e}^{\zeta_i-\zeta_j})\prod_{k=1}^{\tilde{N}}\left(\frac{\mathrm{e}^{-\tilde{y}_k}}{\mathrm{e}^{-\tilde{y}_k} - \mathrm{e}^{-\zeta_j}}\right) = \mathrm{e}^{\nu_{12}\tilde{N}}. \tag{A.13}$$

Defining $Z_j = \mathrm{e}^{-\zeta_j}$ we arrive at

$$\frac{\omega_2(\vec{y})}{\omega_1(\vec{y})}\prod_{i\neq j}\left(-\frac{Z_j}{Z_i}\right)\prod_{k=1}^{\tilde{N}}\left(\frac{\mathrm{e}^{-\tilde{y}_k}}{\mathrm{e}^{-\tilde{y}_k} - Z_j}\right) = \mathrm{e}^{\nu_{12}\tilde{N}} \tag{A.14}$$

and the eigenvalue can be simply written as

$$\Lambda(x) = \mathrm{e}^{\nu_{12}r}\omega_1(\vec{y})\prod_{i=1}^{r}(1-\mathrm{e}^{x}Z_i). \tag{A.15}$$

Appendix B. Yang–Baxter equation for multi-species ASEP with different rates

In this appendix, we want to discuss to what extent the Baxterized form of the R-matrix (12) can be generalized, in order to describe a process with a number of species greater than 3, and a hierarchical structure. Labeling the species with numbers from 1 to n, the hierarchy means that a particle of kind i can hop to the right of a particle of kind j only if $i<j$. It is well known that if all these elementary processes happen with the same rate, then the R-matrix is simply given by the Baxterization of the Hecke algebra [18]. What we want to consider here is the case when the hoppings among the particles depend on the species involved in the hopping

$$ij \to ji \text{ with rate } a_{ij} \qquad \text{if} \quad i<j.$$

Writing the matrix describing the hopping $ij\to ji$ as $E^{(ij)}_{\alpha\beta,\gamma\sigma} = \delta_{\alpha i}\delta_{\beta j}(\mathrm{e}^{\nu_{ij}}\delta_{\alpha\sigma}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\sigma})$, we would like to find an R-matrix of the form

$$R(x,y) = 1 + \sum_{\{ij\}} g_{ij}(x,y)E^{(ij)}, \tag{B.1}$$

which satisfies the Yang–Baxter equation

$$R_{a,b}(y,z)R_{b,c}(x,z)R_{a,b}(x,y) = R_{b,c}(x,y)R_{a,b}(x,z)R_{b,c}(y,z), \tag{B.2}$$

and such that

$$\frac{\mathrm{d}}{\mathrm{d}x} R(x, y)\bigg|_{x=y=0} = \sum_{\{ij\}} a_{ij} E^{(ij)}. \tag{B.3}$$

Let us define for $i < j < k$ the projectors $P_{i,j}$ and $P_{i,j,k}$, which act on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$. $P_{i,j}$ projects on the states occupied only by particles of type i and j, while $P_{i,j,k}$ projects on states occupied only by particles i, j and k. If we intertwine equation (B.2) with $P_{i,j}$ we see that we reduce to the problem with two types of particles (or equivalently one type of particles and the empty sites), and we recover easily that g_{ij} must be of the form

$$g_{i,j}(x, y) = 1 - \frac{f_{i,j}(x)}{f_{i,j}(y)}. \tag{B.4}$$

If we intertwine equation (B.2) with $P_{i,j,k}$ we recover the problem with three types of particles treated in the main body of this paper, and the Yang–Baxter equation implies that

$$f_{i,k}(x) = f_{i,j}(x) + b^{i,j}_{i,k}; \qquad f^{-1}_{j,k}(x) = f^{-1}_{i,k}(x) + b^{i,k}_{j,k}. \tag{B.5}$$

This means that all the functions $f_{i,j}(x)$ are determined in terms of a reference one, which we chose to be $f_{1,2}(x)$, and of the parameters $b^{i,j}_{i,k}$ and $b^{i,k}_{j,k}$. Actually the relations in (B.5) put also constraints on the bs. Indeed it is easy to see that we must have

$$b^{i,j}_{i,k} = \sum_{l=j}^{k-1} b^{i,l}_{i,l+1}; \qquad b^{i,k}_{j,k} = \sum_{l=i}^{j-1} b^{l,k}_{l+1,k}.$$

Moreover if $i > 1$ one can get $f_{i,j+1}(x)$ starting from $f_{i,j}(x)$ in two different ways:

$$f_{i,j} \to f_{i,j+1} \qquad \text{or} \qquad f_{i,j} \to f_{i-1,j} \to f_{i-1,j+1} \to f_{i,j+1}.$$

The previous relation fixes

$$b^{i-1,j+1}_{i,j+1} = \frac{b^{i-1,j}_{i,j}}{1 - b^{i-1,j}_{i-1,j+1} b^{i-1,j}_{i,j}} \qquad \text{and} \qquad b^{i,j}_{i,j+1} = \frac{b^{i-1,j}_{i-1,j+1}}{1 - b^{i-1,j}_{i-1,j+1} b^{i-1,j}_{i,j}}.$$

In conclusion we can chose as free parameters $b^{1,j}_{1,j+1}$ and $b^{l-2,l}_{l-1,l}$, which in a problem with n species are in number of $2(n-1)$. Hence among the $n(n-1)/2$ rates a_{ij}, only $2(n-1)$ are independent, given the form of the R-matrix (B.1). This of course does not rule out the possibility that the problem with generic rates is integrable, but one should look for a more general R-matrix to prove it.

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